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Free skew fields have many $*$-orderings

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Abstract

The notion of a $*$-ordering on a skew-field with involution was introduced by S. Holland in [J. Algebra 101 (1986) 16–46] as an analogue to the notion of a total ordering on a skew-field and developed further in a series of papers of T. Craven, I. Idris, M. Marshall and T. Smith. While it is well known that every free skew-field has a total ordering (see [J. Lewin, Trans. Amer. Math. Soc. 192 (1974) 339–346]), it has not been known so far whether every free skew-field with some natural involution has a $*$-ordering. The aim of this paper is to give an explicit construction of a class of $*$-orderings on free associative algebras and to prove that $*$-orderings from this class extend uniquely to the corresponding free skew fields. The problem was posed by T. Craven and T. Smith in [J. Algebra 238 (2001) 314–327].

Keywords: Free skew fields; Ordered rings; Rings with involution

1. $*$-orderings and $*$-valuations

A *domain* is an associative unital ring without zero divisors. Let R be a domain, Γ an ordered group and $\infty \notin \Gamma$. A mapping $v : R \to \Gamma \cup \{\infty\}$ is a *valuation* if

(1) for every $x \in R$, $v(x) = \infty$ if and only if $x = 0$,
(2) $v(x + y) \geqslant \min\{v(x), v(y)\}$ for every $x, y \in R^\times$,

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(3) $v(xy) = v(x) + v(y)$ for every $x, y \in R^\times$,

where $R^\times := R \setminus \{0\}$. A valuation $v : R \to \Gamma \cup \{\infty\}$ is *quasi-commutative* if $v(xy - yx) > v(x) + v(y)$ for every $x, y \in R$.

A $*$-*domain* is a domain with involution. A valuation v on a $*$-domain R is a $*$-*valuation* if it has an additional property

(4) $v(x^*) = v(x)$ for every $x \in R$.

Note that $v(x) + v(y) = v(xy) = v((xy)^*) = v(y^* x^*) = v(y^*) + v(x^*) = v(y) + v(x)$ for every $*$-valuation v on R and every $x, y \in R$.

For any valuation v on a domain R, we define a relation $\sim_v$ on $R^\times$ by

$$x \sim_v y \quad \Leftrightarrow \quad v(x) < v(x - y).$$

Clearly, $x \sim_v y$ implies that $v(x) = v(y)$. Let us recall Lemma 5.11 from [11]:

Lemma 1. *Let R be a domain and v a valuation on R. The relation $\sim_v$ is an equivalence relation on $R^\times$ and it has the following properties*:

(1) *If $x \sim_v y$, and x, y are invertible, then $x^{-1} \sim y^{-1}$. If $x \sim_v y$ and v is a $*$-valuation, then $x^* \sim_v y^*$.*
(2) *If $x_1 \sim_v y_1$ and $x_2 \sim_v y_2$, then $x_1 x_2 \sim_v y_1 y_2$. If also $x_1 \nsim_v -x_2$, then $x_1 + x_2 \sim_v y_1 + y_2$ as well.*
(3) *If $x \sim_v y$ and $v(z) > v(x)$, then $x + z \sim_v y$.*

Let R be a $*$-domain and $S(R)$ the set of all symmetric elements of R. A subset $P \subseteq S(R)$ is a $*$-*ordering* if

(1) $1 \in P$,
(2) $P + P \subseteq P$,
(3) $rPr^* \subseteq P$ for every $r \in R$,
(4) $S(R) = P \cup -P$,
(5) $P \cap -P = \{0\}$,
(6) if $a, b \in P$, then $ab + ba \in P$.

A $*$-ordering P is *compatible* with a $*$-valuation v if for every $x \in P$ and every $y \in S(R)$ such that $y \sim_v x$ we have that $y \in P$.

Remark. The notation of a $*$-ordering on a skew field was introduced by S.S. Holland in [11] and studied further by T.C. Craven in [3–9]. See [1] for a similar notion. For domains, $*$-orderings were introduced by M. Marshall in [15], see also [10,12,16].

We will need the following proposition in the proof of the main theorem.

Proposition 2. *Let K be a skew field, $v : K \to \mathbb{Z} \cup \{\infty\}$ a quasi-commutative valuation and R a subring of K such that K is generated by R. Then for every $z \in K^\times$ there exist $a, b \in R^\times$ such that $x \sim_v ab^{-1}$.*

Proof. We define recursively a sequence $R_0 \subseteq R_1 \subseteq R_2 \subseteq \cdots$ of subrings of K by $R_0 = R$ and $R_{n+1} = \Sigma(G_n)$, where G_n is the set of all finite products of elements from $R_n \cup R_n^{-1}$ and $\Sigma(G_n)$ is the set of all finite sums of elements from G_n. Note that $\bigcup R_n$ is a subfield of K containing R. Since R generates K, it follows that $K = \bigcup R_n$.

We claim that for every element $z \in R_{n+1}^\times$ there exist elements $a, b \in R_n^\times$ such that $z \sim_v ab^{-1}$. We will prove the claim by constructing a sequence of representations

$$z = u_1^{(k)} + u_2^{(k)} + \cdots + u_{m^{(k)}}^{(k)}$$

(where $k = 0, \ldots, t$ and $u_i^{(k)} \in G_n$ for $i = 1, \ldots, m^{(k)}$) such that the last representation satisfies

$$u_1^{(t)} = a_t b_t^{-1}, \quad a_t, b_t \in R_n^\times \qquad \text{and} \qquad v\big(u_1^{(t)}\big) < \min\big\{v\big(u_2^{(t)}\big), \ldots, v\big(u_{m^{(t)}}^{(t)}\big)\big\}.$$

It follows that $z \sim_v a_t b_t^{-1}$.

By the definition of R_{n+1}, there exist elements $u_1, \ldots, u_m \in G_n$ such that $z = u_1 + \cdots + u_m$. Set

$$m^{(0)} = m \quad \text{and} \quad u_i^{(0)} = u_i \quad \text{for } i = 1, \ldots, m.$$

Assume now that $u_1^{(k-1)}, u_2^{(k-1)}, \ldots, u_{m^{(k-1)}}^{(k-1)}$ have been constructed. Write $m = m^{(k-1)}$. We may assume that

$$v\big(u_1^{(k-1)}\big) = \cdots = v\big(u_p^{(k-1)}\big) < v\big(u_{p+1}^{(k-1)}\big) \leqslant \cdots \leqslant v\big(u_m^{(k-1)}\big).$$

For every $i = 1, \ldots, p$ there exist elements $x_{ij}, y_{ij} \in R_n$ $(j = 1, \ldots, q_i)$ such that

$$u_i^{(k-1)} = \prod_{j=1}^{q_i} x_{ij} y_{ij}^{-1}.$$

Write $x_i = \prod_{j=1}^{q_i} x_{ij}$ and $y_i = \prod_{j=1}^{q_i} y_{ij}$ for every $i = 1, \ldots, p$. Since v is quasi-commutative, we have for every $i = 1, \ldots, p$ that

$$u_i^{(k-1)} = x_i y_i^{-1} + r_i = \left(x_i \prod_{l=1, l \neq i}^{p} y_l\right)\left(\prod_{l=1}^{p} y_l\right)^{-1} + o_i,$$

where o_i and r_i are sums of elements from G_n with value $> v(u_1^{(k-1)})$. Write

$$a_k = \sum_{i=1}^{p}\left(x_i \prod_{l=1,\, l\neq i}^{p} y_l\right) \in R_n \quad \text{and} \quad b_k = \prod_{l=1}^{p} y_l \in R_n^{\times}.$$

It follows that

$$z = a_k b_k^{-1} + \sum_{i=p+1}^{m} u_i + \sum_{i=1}^{p} o_i.$$

Write

$$u_1^{(k)} = a_k b_k^{-1}, \quad u_2^{(k)} = \cdots = u_p^{(k)} = 0, \quad u_{p+1}^{(k)} = u_{p+1}^{(k-1)}, \quad \ldots, \quad u_m^{(k)} = u_m^{(k-1)}$$

and pick $u_{m+1}^{(k)}, \ldots, u_{m^{(k)}}^{(k)} \in G_n$ such that

$$\sum_{i=1}^{p} o_i = \sum_{i=m+1}^{m^{(k)}} u_i^{(k)}.$$

If

$$v\bigl(u_1^{(k)}\bigr) < \min\bigl\{v\bigl(u_2^{(k)}\bigr), \ldots, v\bigl(u_{m^{(k)}}^{(k)}\bigr)\bigr\}$$

then set $t = k$ and stop. Else, repeat the cycle with k replaced by $k+1$.

Write $L_k = \min\{v(u_i^{(k)}), i = 1, \ldots, m^{(k)}\}$. Note that $L_k \leqslant v(z)$ for every $k = 0, \ldots, t$. If $1 < k < t$, then $L_k > L_{k-1}$. It follows that the algorithm terminates in at most $v(z) - L_0$ steps.

By induction on n using the claim and Lemma 1 we prove that for every $z \in R_n$ there exist elements $a, b \in R$ such that $z \sim_v ab^{-1}$. Now, the proposition follows from the fact that $K = \bigcup_n R_n$. $\square$

Note that Proposition 2 implies that Corollary 1 in [14] is true for any Lie algebra L.

Lemma 3. *Let K be a $*$-field and $v : K \to \Gamma \cup \{\infty\}$ a quasi-commutative $*$-valuation on K. For any $s_1, s_2 \in S(K)$ write $\langle s_1, s_2\rangle = \frac{1}{2}(s_1 s_2 + s_2 s_1)$.*

(1) *If $s_1, s_2 \in S(K)^{\times}$, then $\langle s_1, s_2\rangle \neq 0$ and $v(\langle s_1, s_2\rangle) = v(s_1) + v(s_2)$.*
(2) *If $s_1, s_2, t_1, t_2 \in S(K)^{\times}$, $s_1 \sim_v t_1$ and $s_2 \sim_v t_2$ then $\langle s_1, s_2\rangle \sim_v \langle t_1, t_2\rangle$.*
(3) *If $s_1, s_2, s_3 \in S(K)^{\times}$ then $\langle\langle s_1, s_2\rangle, s_3\rangle \sim_v \langle s_1, \langle s_2, s_3\rangle\rangle$.*

Proof. For every $s_1, s_2 \in S(K)^{\times}$, we have that $\langle s_1, s_2\rangle = s_1 s_2 + \frac{1}{2}(s_2 s_1 - s_1 s_2)$. Since v is quasi-commutative, it follows that $v(\frac{1}{2}(s_2 s_1 - s_1 s_2)) > v(s_1 s_2)$. Hence, $v(\langle s_1, s_2\rangle) = v(s_1 s_2)$. It follows that $\langle s_1, s_2\rangle \neq 0$. This proves the first assertion.

To prove the second assertion, note that $\langle s_1, s_2\rangle - \langle t_1, t_2\rangle = \langle s_1, s_2 - t_2\rangle + \langle s_1 - t_1, t_2\rangle$ for any $s_1, s_2, t_1, t_2 \in S(K)$. If $s_1 \sim_v t_1$ and $s_2 \sim_v t_2$ then $v(\langle s_1, s_2\rangle - \langle t_1, t_2\rangle) \geqslant \min\{v(s_1) + v(s_2 - t_2), v(s_1 - t_1) + v(t_2)\} > \min\{v(s_1) + v(s_2), v(t_1) + v(t_2)\} = v(\langle s_1, s_2\rangle) = v(\langle t_1, t_2\rangle)$. Hence, $\langle s_1, s_2\rangle \sim_v \langle t_1, t_2\rangle$.

Since $\langle\langle s_1, s_2\rangle, s_3\rangle - \langle s_1, \langle s_2, s_3\rangle\rangle = \frac{1}{4}(s_2(s_1 s_3 - s_3 s_1) - (s_1 s_3 - s_3 s_1)s_2)$, it follows that $v(\langle\langle s_1, s_2\rangle, s_3\rangle - \langle s_1, \langle s_2, s_3\rangle\rangle) > v(s_1) + v(s_2) + v(s_3) = v(\langle\langle s_1, s_2\rangle, s_3\rangle) = v(\langle s_1, \langle s_2, s_3\rangle\rangle)$. The third assertion follows. □

Theorem 4. *Let K be a $*$-field, $v : K \to \mathbb{Z} \cup \{\infty\}$ a quasi-commutative $*$-valuation on K and R a $*$-subring of K such that K is generated by R. Then every $*$-ordering on R which is compatible with $v|_R$ extends uniquely to a $*$-ordering on K compatible with v.*

Proof. We claim that for every $z \in S(K)^\times$, there exist elements $s, t \in S(R)^\times$ such that $z \sim_v \langle s, t^{-1}\rangle$. By Proposition 2, there exist elements $a, b \in R^\times$ such that $z \sim_v ab^{-1}$. Write $c = ab^*$ and $t = bb^* \in S(R)^\times$ and note that $ab^{-1} = ct^{-1}$. We have $ct^{-1} \sim_v t^{-1}c^*$ since $z = z^*$ and $ct^{-1} \sim_v t^{-1}c$ since v is quasi-commutative. It follows that $c \sim_v c^*$. Write $s = \frac{1}{2}(c + c^*) \in S(R)^\times$ and note that $z \sim_v ct^{-1} \sim_v st^{-1} \sim_v \langle s, t^{-1}\rangle$. This proves the claim.

Let P' be a v-compatible $*$-ordering on K. Then $P := P' \cap S(R)$ is a $v|_R$-compatible $*$-ordering on R. We claim that

$$P' = \left\{z \in S(K) \,\middle|\, \exists s, t \in P\colon\ z \sim_v \left\langle s, t^{-1}\right\rangle\right\}.$$

If $z \in (P')^\times$, then there exist $s, t \in S(R)^\times$ such that $z \sim_v \langle s, t^{-1}\rangle$. Changing signs if necessary, we may assume that $t \in P$. Since P' is v-compatible and $s \sim_v \langle\langle s, t^{-1}\rangle, t\rangle \sim_v \langle z, t\rangle \in P'$ by Lemma 3, it follows that $s \in P' \cap S(R) = P$. So, z belongs to the set on the right side. The other inclusion is trivial. This proves uniqueness.

To prove existence, take any $v|_R$-compatible $*$-ordering P on R and define P' as above. We claim that P' is a v-compatible $*$-ordering which extends P.

Compatibility follows directly from the definition. Clearly, $P \subseteq P' \cap S(R)$. Conversely, if $x \in P' \cap S(R)$, then there exist elements $s, t \in P$ such that $x \sim_v \langle s, t^{-1}\rangle$. Since P is v-compatible and $\langle x, t\rangle \sim_v s$, it follows that $\langle x, t\rangle \in P$. If $x \in -P$, then $\langle x, t\rangle \in P \cap -P = \{0\}$, so that $x = 0$. It follows that $x \in P$. Hence, $P = P' \cap S(R)$. We have to verify that P' satisfies properties (1)–(6) from the definition of a $*$-ordering.

Clearly, $1 \in P'$. If $x, y \in (P')^\times$ then there exist $s_1, s_2, t_1, t_2 \in P^\times$ such that $x \sim_v \langle s_1, t_1^{-1}\rangle$ and $y \sim_v \langle s_2, t_2^{-1}\rangle$. It follows that $\langle x, \langle t_1, t_2\rangle\rangle \sim_v \langle s_1, t_2\rangle \in P$ and $\langle y, \langle t_1, t_2\rangle\rangle \sim_v \langle s_2, t_1\rangle \in P$. Since $\langle s_1, t_2\rangle \nsim_v -\langle s_2, t_1\rangle$, it follows that $\langle x + y, \langle t_1, t_2\rangle\rangle \sim_v \langle s_1, t_2\rangle + \langle s_2, t_1\rangle \in P$ by assertion (2) of Lemma 1. Hence, $x + y \in P'$.

If $x \in (P')^\times$ and $d \in K^\times$, then there exist $s, t \in S(R)^\times$ and $a, b \in R^\times$ such that $x \sim_v \langle s, t^{-1}\rangle$ and $d \sim_v ab^{-1}$. By assertion (1) of Lemma 1 we have that $d^* \sim_v (b^*)^{-1}a^*$. It follows that $dxd^* \sim_v \langle asa^*, (btb^*)^{-1}\rangle$. Since $asa^* \in P$ and $btb^* \in P$, it follows that $dxd^* \in P'$.

If $x \in S(K)^\times$, then by the claim in the first paragraph of this proof there exist $s, t \in S(R)^\times$ such that $x \sim_v \langle s, t^{-1}\rangle$. If s and t have the same sign with respect to P, then $x \in P'$, otherwise $x \in -P'$.

If $x \in P' \cap -P'$ is nonzero, then there exist $s_1, s_2, t_1, t_2 \in P^\times$ such that $x \sim_v \langle s_1, t_1^{-1}\rangle$ and $-x \sim_v \langle s_2, t_2^{-1}\rangle$. It follows that $\langle x, \langle t_1, t_2\rangle\rangle \sim_v \langle s_1, t_2\rangle \in P$ and $\langle -x, \langle t_1, t_2\rangle\rangle \sim_v \langle s_2, t_1\rangle \in P$. It follows that $\langle s_1, t_2\rangle \sim_v -\langle s_2, t_1\rangle$, a contradiction with $P \cap -P = \{0\}$.

If $x, y \in (P')^\times$ and $x \sim_v \langle s_1, t_1^{-1}\rangle$, $y \sim_v \langle s_2, t_2^{-1}\rangle$, then $\langle\langle x, y\rangle, \langle t_1, t_2\rangle\rangle \sim_v \langle\langle x, t_1\rangle, \langle y, t_2\rangle\rangle \sim_v \langle s_1, s_2\rangle \in P$. Hence $\langle x, y\rangle \in P'$. □

2. Enveloping algebras and their skew-fields of fractions

Let L be a complex Lie algebra and $U(L)$ its enveloping algebra. In [13] Lichtman gives a construction of a skew field $D(L)$ which contains $U(L)$ and is generated by it. We recall the basic steps of the construction for later reference.

The *standard filtration* $\cdots \subset F_1 \subset F_0 \subset F_{-1} \subseteq \cdots$ of $U(L)$ is defined by $F_i = \{0\}$ if $i > 0$, $F_0 = \mathbb{C}$ and for $i = -k < 0$, F_i is a vector subspace of $U(L)$ generated by all products containing $\leqslant k$ elements from L. The mapping $v : U(L) \to \mathbb{Z} \cup \{+\infty\}$ defined by $v(x) = \sup\{n \mid x \in F_n\}$ is a valuation called the *standard valuation*. Let $U(L)[t, t^{-1}]$ be the ring of Laurent polynomials in a central variable t. Extend v to a valuation on $U(L)[t, t^{-1}]$ by $v(\sum c_i t^i) = \min\{v(c_i) + i\}$. Write $R := \{x \in U(L)[t, t^{-1}] \mid v(x) \geqslant 0\} \subset U(L)[t]$. Note that $U(L)[t, t^{-1}]$ is canonically isomorphic to the localization $R[1/t]$ and that the valuation v on $U(L)[t, t^{-1}]$ corresponds to the t-adic valuation on $R[1/t]$. For every $n = 1, 2, \ldots$ write $R_n = R/t^n R$ and U_n for the projection of $U = R \setminus tR$ in R_n. It turns out that U_n is a regular Ore set in R_n for every n and we denote by $S_n = (R_n)_{U_n}$ its Ore localization. Write S for the inverse limit of S_n. The localization $D = S[1/t]$ is a skew field. There exists a natural embedding of R into S which extends to a natural embedding of $U(L)[t, t^{-1}] = R[1/t]$ into $D = S[1/t]$. Write $D(L)$ for the minimal subfield of D containing $U(L)$. The restriction of the t-adic valuation from $D = S[1/t]$ to $D(L)$ will be called the *standard valuation* on $D(L)$. It extends the standard valuation on $U(L)$ and it is known to be quasi-commutative.

Let L be a complex Lie algebra. A mapping $* : L \to L$ is an *involution* if for all $x, y \in L$ and $c \in \mathbb{C}$, $(x + y)^* = x^* + y^*$, $(cx)^* = \bar{c}x^*$, $[x, y]^* = [y^*, x^*]$, $x^{**} = x$.

Proposition 5. *Every involution on L has a canonical extension to $D(L)$ such that the standard valuation on $D(L)$ is a $*$-valuation.*

Proof. Every involution extends uniquely from L to its enveloping algebra $U(L)$ and the standard valuation on $U(L)$ is a $*$-valuation. Setting $t^* = t$ we get an involution on $U(L)[t, t^{-1}]$ which induces an involution on R. Since $(t^n R)^* \subseteq t^n R$ for every n, we have an induced involution on $R/t^n R$ which will also be denoted by $*$. Note that the natural epimorphisms $\phi_n : R_{n+1} \to R_n$ are $*$-homomorphisms. The same proof as in [10, Lemma 2.2] shows that the involution on R_n extends uniquely to an involution of $S_n = (R_n)_{U_n}$. It is easy to verify that the natural epimorphisms $\phi'_n : S_{n+1} \to S_n$ are $*$-homomorphisms. It follows that the termwise involution on the inverse system of S_n and ϕ'_n induces an involution on its inverse limit S. Since $t^* = t$, this involution extends uniquely to an involution of $D = S[1/t]$. It is clear that the t-adic valuation on $S[1/t]$ is a $*$-valuation which extends the t-adic valuation on $R[1/t]$. Since $D(L) \cap D(L)^*$ is a subfield of D containing

$U(L) = U(L)^*$ and since $D(L)$ is the smallest subfield of D containing $U(L)$, it follows that $D(L) = D(L)^*$. Hence, $D(L)$ is a $*$-subfield of D. $\square$

Any involution on a complex Lie algebra L preserves the standard filtration of $U(L)$ and induces an involution on the corresponding graded ring $\bigoplus_i F_i/F_{i+1}$ which is isomorphic to the complex polynomial ring in $\dim L$ variables by the Poincare–Birkhoff–Witt Theorem.

Theorem 6. *Let L be a complex Lie algebra with involution. There exists a canonical* 1–1–1 *correspondence between*

(1) *$*$-orderings on $D(L)$ compatible with the standard valuation,*
(2) *$*$-orderings on $U(L)$ compatible with the standard valuation,*
(3) *$*$-orderings on the complex polynomial ring in* $\dim L$ *variables compatible with the total degree.*

Proof. The standard valuation v on $D(L)$ is a quasi-commutative $*$-valuation. Hence the 1–1 correspondence between $*$-orderings in (1) and (2) follows from Theorem 4. The 1–1 correspondence between $*$-orderings in (2) and (3) follows from Proposition 2.5 in [16]. $\square$

If L is a finite dimensional Lie algebra, then $U(L)$ is an Ore domain and $D(L)$ is its skew field of fractions. Hence, Theorem 6 is a special case of Corollary 2.5 in [10] and Corollary 4.3 in [16].

Let $L_{\mathcal{S}}$ be the free Lie algebra on a set $\mathcal{S}$, then its enveloping algebra $U(L_{\mathcal{S}})$ is isomorphic to the free associative algebra $\mathbb{C}\langle\mathcal{S}\rangle$ on $\mathcal{S}$ (Theorem 0.5 in [17] or a remark after Theorem 2.6.6 in [2]) and the field $D(L_{\mathcal{S}})$ is isomorphic to the free field $\Delta_{\mathcal{S}}$ on $\mathcal{S}$ (Theorem 1 in [14]). Note that the group ring $\mathbb{C}[F_{\mathcal{S}}]$ of the free group $F_{\mathcal{S}}$ on $\mathcal{S}$ contains $\mathbb{C}\langle\mathcal{S}\rangle$ and is contained in $\Delta_{\mathcal{S}}$.

We can define an involution on $L_{\mathcal{S}}$ by $x^* = -x$ for every $x \in \mathcal{S}$ and $c^* = \bar{c}$ for every $c \in \mathbb{C}$. This involution extends to $\mathbb{C}\langle\mathcal{S}\rangle$, $\mathbb{C}[F_{\mathcal{S}}]$ and $\Delta_{\mathcal{S}}$ by Proposition 5. We have the following corollary of Theorem 6.

Corollary 7. *Let $\mathcal{S}$ be an arbitrary set, $L_{\mathcal{S}}$ the free Lie algebra on $\mathcal{S}$ with the involution defined above. The standard valuation v on $\Delta_{\mathcal{S}} = D(L_{\mathcal{S}})$ is a $*$-valuation. We have a canonical* 1–1–1–1 *correspondence between*

(1) *$*$-orderings on $\Delta_{\mathcal{S}}$ compatible with v,*
(2) *$*$-orderings on $\mathbb{C}[F_{\mathcal{S}}]$ compatible with v,*
(3) *$*$-orderings on $\mathbb{C}\langle\mathcal{S}\rangle$ compatible with v,*
(4) *$*$-orderings on the complex polynomial ring in* $\dim L_{\mathcal{S}}$ *variables compatible with the total degree.*

The following example solves a problem posed by T. Craven and T. Smith in [10].

Example. Let the involution on $A = \mathbb{Z}\langle a, b\rangle$ be defined by $a^* = b$ and $b^* = a$. By Example 3.2. in [10], we know that $*$-orderings on A exist. We are able to give an explicit description of a class of $*$-ordering on A. Let the involution on $A' = \mathbb{C}\langle a_1, a_2\rangle$ be defined by $a_1^* = -a_1$ and $a_2^* = -a_2$ and $c^* = \bar{c}$ for every $c \in \mathbb{C}$. The homomorphism $\phi : A \to A'$ defined by $\phi(a) = \frac{1}{2}(a_1 + ia_2)$, $\phi(b) = \frac{1}{2}(-a_1 + ia_2)$ satisfies $\phi(a^*) = \phi(a)^*$ and $\phi(b^*) = \phi(b)^*$, hence it is an embedding of $*$-rings. For any $*$-ordering P on A' the set $\phi^{-1}(P)$ is a $*$-ordering on A. Corollary 7 reduces the construction of $*$-orderings on A' to the construction of $*$-orderings on the complex polynomial ring $\mathbb{C}[x_k]_{k\in\mathbb{N}}$. Replacing variables x_k with $y_k = ix_k$, we get $S(\mathbb{C}[x_k]_{k\in\mathbb{N}}) = \mathbb{R}[y_k]_{k\in\mathbb{N}}$. Hence, there is a natural 1–1 correspondence between $*$-orderings on $\mathbb{C}[x_k]_{k\in\mathbb{N}}$ and total orderings on $\mathbb{R}[y_k]_{k\in\mathbb{N}}$.

The following example shows that there exist $*$-orderings on $\mathbb{C}\langle a, b\rangle$ which are not compatible with the standard valuation

Example. Let $R = \mathbb{C}\langle a, b\rangle$ with involution $*$ defined by $a^* = -a$, $b^* = -b$ and $c^* = \bar{c}$ for $c \in \mathbb{C}$. Let v be the standard valuation on R and P a $*$-ordering on R compatible with v. Let $\deg : R \to \mathbb{Z} \cup \{-\infty\}$ be the total degree function in a and b. For every element $x \in R$ write $x = \operatorname{lt}(x) + o$, where $\operatorname{lt}(x)$ is a sum of all terms in x of total degree $\deg(x)$ and $\deg(o) < \deg(x)$. Write $Q = \{x \in S(R) \mid \operatorname{lt}(x) \in P\}$. We claim that Q is a $*$-ordering on R which is not compatible with v. We must verify properties (1)–(6) of $*$-orderings.

Clearly, $1 \in Q$. Take any $x, y \in Q$. Note that $\operatorname{lt}(x)$ and $\operatorname{lt}(y)$ cannot cancel each other, since they both belong to P. If $\deg(x) = \deg(y)$, then $\operatorname{lt}(x + y) = \operatorname{lt}(x) + \operatorname{lt}(y) \in P$. If $\deg(x) \neq \deg(y)$, then $\operatorname{lt}(x + y)$ is either $\operatorname{lt}(x)$ or $\operatorname{lt}(y)$. In all cases $\operatorname{lt}(x + y) \in P$, so that $x + y \in Q$. If $x \in Q$ and $r \in R$, then $\operatorname{lt}(rxr^*) = \operatorname{lt}(r)\operatorname{lt}(x)\operatorname{lt}(r)^* \in \operatorname{lt}(r)P\operatorname{lt}(r)^* \subseteq P$, so that $rxr^* \in Q$. For every $x \in S(R)$, $\operatorname{lt}(x) \in P \cup -P$, hence $x \in Q \cup -Q$. If $x \in Q \cap -Q$, then $\operatorname{lt}(x) \in P \cap -P = \{0\}$. If follows that $x = 0$. If $x, y \in Q$, then $\operatorname{lt}(xy) = \operatorname{lt}(x)\operatorname{lt}(y)$ and $\operatorname{lt}(yx) = \operatorname{lt}(y)\operatorname{lt}(x)$ cannot cancel each other. Hence $\operatorname{lt}(xy + yx) = \operatorname{lt}(x)\operatorname{lt}(y) + \operatorname{lt}(y)\operatorname{lt}(x) \in P$, so that $xy + yx \in Q$.

Finally Q is not v-compatible. Take $x = [a, b]^2$ and $y = i[a, [a, [a, [a, b]]]]$. We have $x, y \in S(R)$, $v(x) = -2 < -1 = v(y)$ and $\deg(x) = 4 < 5 = \deg(y)$. Changing signs if necessary, we may assume that $x, y \in P$. It follows that $\operatorname{lt}(y - x) = y \in P$, so that $y - x \in Q$. If Q were compatible with v, then $x - y \in Q$, a contradiction.

3. Open problems

(1) Do $*$-orderings on a free algebra which are not compatible with the standard valuation extend to the corresponding free field? Is the extension unique?
(2) Do $*$-orderings always extend from $*$-domains to their universal fields of fractions.
(3) Is every $*$-domain which has a $*$-ordering embedable in a $*$-field? If not find nontrivial sufficient conditions for embedability.

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Further reading

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